Special Functions in Physics Toolbox

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- More than 170 special functions are supported. From A for Anger function to Z for Zernike polynomial.
- The function can be evaluated in the complex domain.
- Noninteger, complex indices are supported, if appropriate.
- The documentation is searchable and embedded in the MATLAB Help Browser.
- The toolbox is based on MATLAB Rel. 2018b.

Special functions play an important role in physics, chemistry, mathematics and engineering. The Special Functions in Physics Toolbox (SPECFUNPHYS toolbox) comes with more than 170 special functions in the complex domain, covering in addition non-integer indices where appropriate. The source codes are readable and not write protected, thus can be modified by the reader. If this should be necessary, I recommend first to copy the SPECFUNPHYS function under an new name and to modify this copy, because other SPECFUNPHYS programs might depend on the function in quest. E.g., the SPECFUNPHYS gamma function, $\Gamma(x)$, covers as well the complex domain and will be called by many other SPECFUNPHYS functions. The toolbox should be part of the MATLAB search path, because the SPECFUNPHYS functions live in different folders. The SPECFUNPHYS toolbox comes in addition with a detailed, searchable html-based documentation.

By double-clicking onto the file SpecFunPhys.mltbx the SPECFUNPHYS toolbox will be installed. Path settings are automatically adjusted as needed. On the MATLAB-documentation a `Supplemental Software` link appears at the bottom of the documentation home page. Clicking this link opens the documentation with links to the Content Guide, an overview of the content, and a link to all supported special functions. From this two links one is directly guided to the html-description of the various toolbox functions respectively classes. The documentation is searchable due to an additional search database. Thus entering a keyword in the search box (Search Help) will list all hits. The html-based function description can automatically opened in the MATLAB documentation window via >> web function_name.html, e.g., >> `web gammaC.html`. (`gammaC` is the name of the SPECFUNPHYS Γ-function.) This html-files are for convenience also added as complementary m-files with the additional name doc. Therefore, e.g., >> `edit gammaCdoc` will open this m-files for direct access of the examples related to the SpecFunPhys class `gammaC`. The SpecFunPhys toolbox is based on Matlab version R2018b and tested under Windows and Linux operating systems. No other Matlab toolbox is necessary for using this toolbox. The SpecFunPhys toolbox is related to the book W. Schweizer: Special Functions in Physics with Matlab, Springer Nature Switzerland AG 2021

Content Guide

Gamma functions, Beta functions and related functions Covers Gamma functions, Beta functions and related functions, e.g., Pochhammer symbols in the complex domain. The computation of the gamma function is based on its logarithm. Thus in addition the logarithm of the functions as well as the incomplete versions are supported.

Error Functions and Fresnel Integrals Covers the Error function, Fresnel integrals and related functionalities like the Voigt Profile functions.

Legendre Polynomials and Legendre Functions Topic are the Legendre Polynomials and Legendre Functions of first and second kind. The various functions or classes support both the complex domain with respect to the argument and with respect to the indices. Legendre Polynomials and Legendre Functions play an important role in many areas of physics.

Bessel and Airy Functions Bessel and Airy functions are in many areas of physics used. The toolbox functions evaluate in addition modulus and phase of the Airy functions, the Scorer functions and, e.g., the Kelvin functions for complex indices.

Struve Functions and Related Functions The Struve, Weber, and Anger functions are solutions of the generalized Bessel differential equation.

Confluent Hypergeometric Function Hypergeometric functions are of importance for many numerical applications in physics. The class conhyp supports the evaluation of the confluent hypergeometric function of first and second kind, the evaluation of the Whittaker function, the confluent hypergeometric limit function and the function ${}_2F_0(z)$.

Coulomb Wave Functions Evaluation of the regular and irregular Coulomb functions and scattering states.

Gauss Hypergeometric Functions play an important in role in many area of mathematical physics and numerical evaluations.

Theta Functions Jacobi or classical theta function play an important role in evaluating modular and elliptic functions. Additional topics are Dedekind's Eta function and the Jacobi index.

Jacobi Elliptic Functions Jacobi Elliptic Functions appear in a variety of applications in engineering and physics, e.g., in hydrodynamics, general relativity, or classical dynamics.

Elliptic Integrals Evaluation of the complete and incomplete elliptic integrals of first, second and third kind in the complex domain for various representations (Jacobi, Carlson, Bulirsch), and some related functions.

Weierstrass Functions Evaluation of the Weierstrass elliptic functions and some related functions.

Parabolic Cylinder Functions are related to the solution of the Laplace Beltrami operator or Helmholtz equation in parabolic cylinder coordinates.

Mathieu Functions are related to the solutions of the Helmholtz equation in elliptic cylindrical coordinates. Applications are, e.g., related to the dynamical trapping of particles in a Paul trap. Additionally the solutions of the modified or radial Mathieu equations are evaluated.

Orthogonal Polynomials Orthogonal polynomials play an important role in physics. E.g., Jacobi and Gegenbauer polynomials are evaluated as well as some general methods developed to support various computations with orthogonal polynomials.

Hermite Polynomials are related to the eigenfunctions of the quantum harmonic oscillator. Thus play an important role, especially in quantum dynamics.

Laguerre Polynomials play an important role in solving the radial Schrödinger equation for Coulomb systems. Here we evaluate generalized or associate Laguerre polynomials.

Chebychev Polynomials are orthogonal polynomials. Applications are, e.g., in approximation or in constructing wavelets. Supported are Chebychev polynomials and arbitrarily shifted Chebychev polynomials of 1st, 2nd, 3rd, and 4th kind in the complex domain and for non-integer indices (Chebychev functions).

Bernoulli and Euler Polynomials and the corresponding numbers are frequently used in statistical physics.

Riemann Zeta Function Target are the Riemann Zeta function and its non-trivial zeros. Applications can be found, e.g., in quantum theory or string theory.

Piecewise Interpolation Polynomials are useful for interpolation techniques, e.g., for finite elements (example for the Hydrogen atom included). The programs allow to compute the interpolation polynomial coefficients for Lagrange interpolation polynomials, Hermite interpolation polynomials and extended Hermite interpolation polynomials of arbitrary degree.

Wigner- and Clebsch-Gordan Coefficients Angular momentum and their coupling are an important concept for quantum systems. Topics are Clebsch-Gordan coefficients, Wigner 3j-symbols, Wigner 6j-symbols, and Wigner 9j-symbols.

Coordinate Systems play an important role in treating physical systems. E.g., the three dimensional Laplace-Beltrami operator is separable in 11 curvilinear coordinates. Under this title transformation equations between various coordinate systems are programmed. E.g., between all 11 curvilinear coordinates mentioned above and other frequently used coordinate systems in physics.

Alphabetic list of supported special functions

A
Airy function $Ai(z)$ and $Bi(z)$
Anger function $J_\nu(z)$
B
Bernoulli polynomials $B_n(x)$, Bernoulli numbers B_n
Beta function $B(z,w)$, Beta function, incomplete $B_x(z,w)$
Bessel function, first kind $J_\nu(z)$, Bessel function, modified 1st kind $I_\nu(z)$
Bessel function, modified 2nd kind $K_\nu(z)$, Bessel function, second kind $Y_\nu(z)$
Bessel function, spherical 1st kind $j_n(z)$, 2nd kind $y_n(z)$ and 3rd kind $h_m^{(n)}(z), n \in \{1,2\}$
Bessel function, spherical $i_m^{(n)}(z), n \in \{1,2\}$, Bessel function, spherical $k_n(z)$
Bessel function, third kind $H_\nu^{(n)}(z), n \in \{1,2\}$
Burlirsch's elliptic integrals $el1(x,kc), el2(x,kc,a,b), el3(x,kc,a,b), cel(kc,p,a,b)$
C
Carlson's elliptic function $RC(x,y)$
Carlson's elliptic integral of 1st kind $RF(x,y,z)$
Carlson's elliptic integral of 2nd kind $RG(x,y,z), RD(x,y,z)$
Carlson's elliptic integral of 3rd kind $RJ(x,y,z,p)$
Chebychev polynomial 1st kind, shifted $T_n(x), T_n^*(x)$
Chebychev polynomial 2nd kind, shifted $U_n(x), U_n^*(x)$
Chebychev polynomial 3rd kind, shifted $V_n(x), V_n^*(x)$
Chebychev polynomial 4th kind, shifted $W_n(x), W_n^*(x)$
Clebsch-Gordan coefficient $C(\cdot j \cdot ; \cdot m \cdot)$
Coherent states $|\alpha\rangle$ (quantum dynamics)
Confluent hypergeometric function, 1st kind ${}_1F_1(a,b,z)$, 2nd kind $U(a,b,z)$
Confluent hypergeometric limit function ${}_0F_1(a,b,z)$
Conical function, 1st kind $P^\mu_{-\frac{1}{2}+i\tau}(z)$ and 2nd kind $Q^\mu_{-\frac{1}{2}+i\tau}(z)$
Coulomb wave function, irregular $H_l^\pm(\gamma,\rho)$ and $h_l(\varepsilon,\zeta)$
Coulomb wave function, irregular $G_l^\pm(\gamma,\rho)$ and regular $F_l^\pm(\gamma,\rho)$
Coulomb wave function, regular $f_l^\pm(\varepsilon,\zeta)$
Coulomb wave function, parabolic $\psi_k(\vec{r})$
Cylinder function $J_\nu(z)$
D
Dawson integral $F(z)$
Dedekind's η Function $\eta(\tau)$
Digamma function $\psi(z)$
Discriminant $\Delta(\Omega_1,\Omega_3)$

E
Elliptic integral of first kind $K(k), F(\phi, k)$, of second kind $E(k), E(\phi, k)$
Elliptic integral of third kind $\Pi(n,k), \Pi(\phi,n,k)$ and symmetric Elliptic integral $D(\phi,k)$
Elliptic modular function $\lambda(\tau)$
Error function $\mathrm{erf}(z)$ and complementary Error function $\mathrm{erfc}(z)$
Euler polynomial $E_n(x)$ and Euler number E_n
F
Fresnel integral $F_F(z)$, Fresnel integral, cosine $C(z)$, Fresnel integral, sine $S(z)$
G
Gamma function $\Gamma(z)$, incomplete Gamma function $\Gamma(z,a)$
Gamma function, inverse incomplete
Gauss hypergeometric function ${}_2F_1(a,b;c;z)$
Gegenbauer polynomial $C_n^{\lambda}(x)$
H
Hankel function $H_{\nu}^{(n)}(z), n \in \{1,2\}$
Harmonic oscillator states $\psi_n(x)$ (quantum mechanics)
Hermite interpolation polynomial $\Phi(x), \bar{\Phi}(x)$
Hermite interpolation polynomial, extended $\Phi(x), \bar{\Phi}(x), \bar{\bar{\Phi}}(x)$
Hermite polynomials $H_n(x)$
Heumann's Lambda function $\Lambda_0(\beta,k)$
Hypergeometric function ${}_2F_0(a,b,z)$, Hypergeometric function ${}_2F_1(a,b;c;z)$
J
Jacobi elliptic function $sn(z,k), ns(z,k)$, $cn(z,k), nc(z,k)$, $dn(z,k), nd(z,k)$
Jacobi elliptic function $sd(z,k), ds(z,k)$, $cd(z,k), dc(z,k)$ $sc(z,k), cs(z,k)$ and $am(z,k)$
Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$
Jacobi symbol $\left(\frac{a}{b}\right)$
Jacobi ϑ functions $\vartheta_n(z,\tau), n \in \{1,2,3,4\}$
Jacobi zeta function $Z(\beta,k)$
K
Kelvin functions $bei_{\nu}(z), ber_{\nu}(z)$, $kei_{\nu}(z), ker_{\nu}(z)$
Klein's complete invariant $J(\tau)$
L
Lagrange interpolation polynomial $\Phi(x)$
Laguerre polynomial $L_n^{\alpha}(x)$
Lattice invariants $g2(\Omega_1,\Omega_3), g3(\Omega_1,\Omega_3)$
Lattice roots $e_i(\Omega_1,\Omega_3), i \in \{1,2,3\}$
Legendre function, 2nd kind $Q_n(x)$ and $Q_l^m(x)$
Legendre function, 2nd kind $Q_{\nu}^{\mu}(z)$ (complex indices)
Legendre function, associate $P_l^m(x)$ and with complex indices $P_{\nu}^{\mu}(z)$
Legendre elliptic integral of first kind $K(k), F(\phi,k)$ and of second kind $E(k), E(\phi,k)$
Legendere elliptic integral of third kind $\Pi(n,k), \Pi(\phi,n,k)$
Legendre polynomial $P_n(x)$
M
Mathieu function $ce_{\nu}(q,z), se_{\nu}(q,z)$ and $me_{\nu}(q,z)$
Mathieu function, 2nd solution $fe_n(q,z), ge_n(q,z)$

Modified Mathieu function $Ce_\nu(q,z), Se_\nu(q,z)$ and $Me_\nu(q,z)$
Modified Mathieu function, 2nd solution $Fe_n(q,z), Ge_n(q,z)$
Mehler function, 1st kind $P^\mu_{-\frac{1}{2}+i\tau}(z)$ and 2nd kind $Q^\mu_{-\frac{1}{2}+i\tau}(z)$

N

Neumann function $Y_\nu(z)$

O

Oblate spheroidal function $R_n^m(x)$ and $T_n^m(x)$

P

Pochhammer Symbol $(z)_w$
Psi Function $\psi(z)$

R

Riemann zeta function $\zeta(s)$
Ring function, 1st kind $P^\mu_{-\frac{1}{2}+\tau}(z)$ and 2nd kind $Q^\mu_{-\frac{1}{2}+\tau}(z)$

S

Scorer function $Gi(z)$ and $Hi(z)$
Spherical harmonics $Y_l^m(\theta,\phi)$
Struve function $H_\nu(z)$ and $K_\nu(z)$

T

ϑ functions $\vartheta_n(z,\tau), n \in \{1,2,3,4\}$
Toroidal function, 1st kind $P^\mu_{-\frac{1}{2}+\tau}(z)$ and 2nd kind $Q^\mu_{-\frac{1}{2}+\tau}(z)$

V

Voigt profile $U(x,t)$ and $V(x,t)$
Voigt profile, line broadening $H(a,u)$

W

Wave function, radial $R_{n,l}(r)$ (quantum mechanics)
Weber function $E_\nu(z)$
Weierstraß elliptic functions $\wp(z), \sigma(z), \zeta(z)$
Whittaker function $M_{a,b}(z)$ and $W_{a,b}(z)$
Wigner function $W(q,p)$
Wigner rotation function $d^l_{m,m'}(\beta)$
Wigner 3j-symbol, 6j-symbol and 9j-symbol

Z

Zernike polynomial $Z_n^m(\rho,\phi)$